Math 4650

Topic 1 - Properties of the real numbers

Def: A **field** is a set F with two operations, addition and multiplication, such that:

(A1) If $x, y \in F$, then $x + y \in F$
(A2) For all $x, y \in F$, we have $x + y = y + x$
(A3) For all $x, y, z \in F$, we have $x + (y + z) = (x + y) + z$
(A4) F contains an element 0 where
$0 + x = 0$ for all $x \in F$.
(A5) For every $x \in F$ there exists an element
$-x \in F$ where $x + (-x) = 0$.

(M1) If $x, y \in F$, then $xy \in F$.
(M2) If $x, y \in F$, then $xy = yx$
(M3) If $x, y, z \in F$, then $x(yz) = (xy)z$
(M4) F contains an element 1 where $1 \neq 0$
and $1x = x$ for all $x \in F$
(M5) If $x \in F$ and $x \neq 0$ then there exists
an element $x^{-1} \in F$ where $x x^{-1} = 1$.

(D1) If $x, y, z \in F$, then $x(y + z) = xy + xz$

Def: An <u>ordered field</u> is a field F along with a relation $<$ on F where

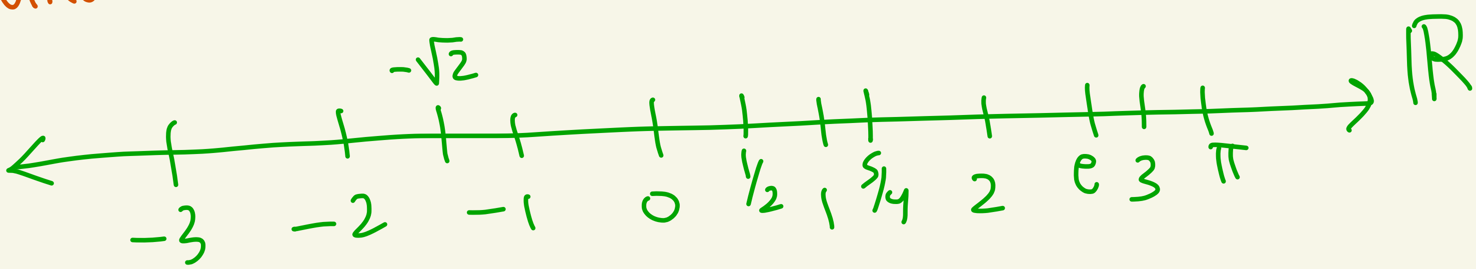
(O1) If $x, y \in F$, then one and only one of the following is true:
$x<y$, $x=y$, $y<x$

(O2) If $x,y,z \in F$ with $x<y$ and $y<z$, then $x<z$.

(O3) If $x,y,z\in F$ and $y<z$, then $x+y<x+z$

(O4) If $x,y \in F$ with $x>0$ and $y>0$, then $xy>0$.

<u>Assumption:</u> We will assume that the set of real numbers $\mathbb{R}$ exists and that it is an ordered field.



From the ordered field properties

one is able to derive all the usual algebraic and order properties of $\mathbb{R}$ that you are used to.

We will assume all these usual properties.

If you want to see how to derive then see the optional Topic 1a notes on the website.

If there is time at the end of the semester I will show you how to construct $\mathbb{R}$ from $\mathbb{Q}$ using "Dedekind cuts". We can then derive the field and order properties.

<u>Def:</u> (Interval notation)

$(a,b) = \{x \mid x \in \mathbb{R},\ a < x < b\}$

$[a,b] = \{x \mid x \in \mathbb{R},\ a \le x \le b\}$

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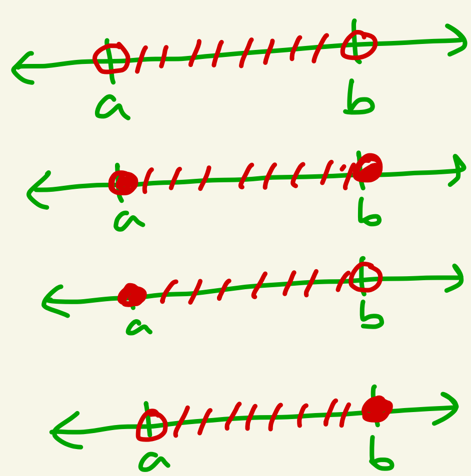


We will also assume the following subsets of $\mathbb{R}$ have their usual algebraic/order properties

set of natural numbers:
$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

set of integers:
$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

set of rational numbers:
$$\mathbb{Q} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z},\ n \neq 0 \right\}$$

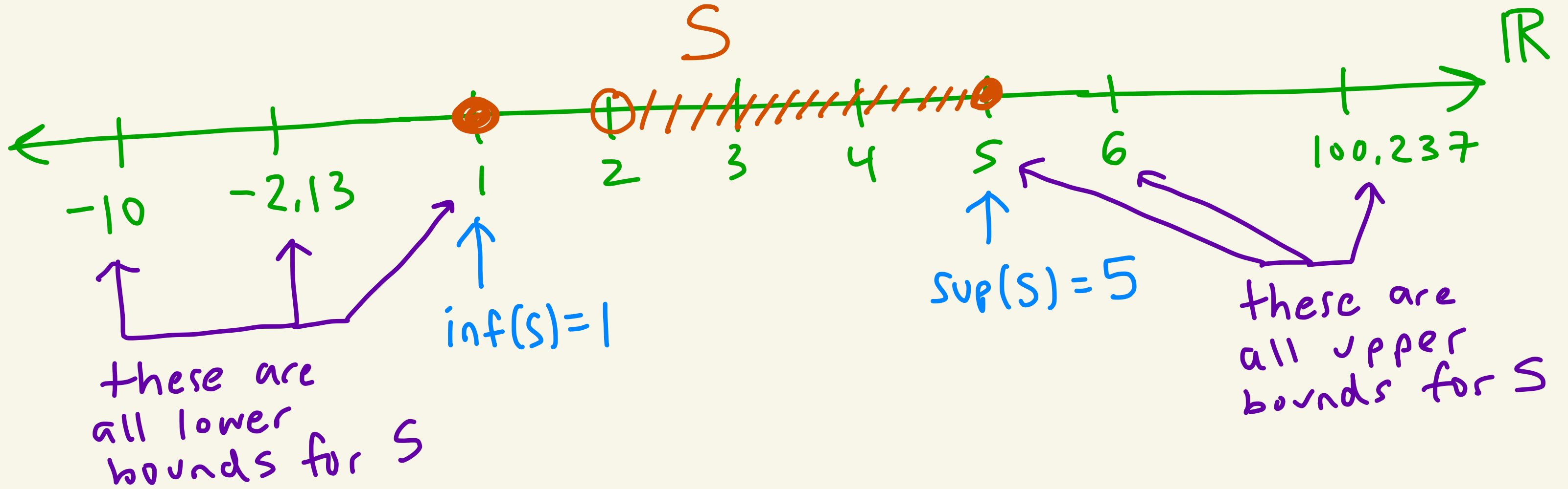
There are some other assumptions we will make, namely that $\mathbb{R}$ satisfies the "completeness axiom"

Let's develop this next.

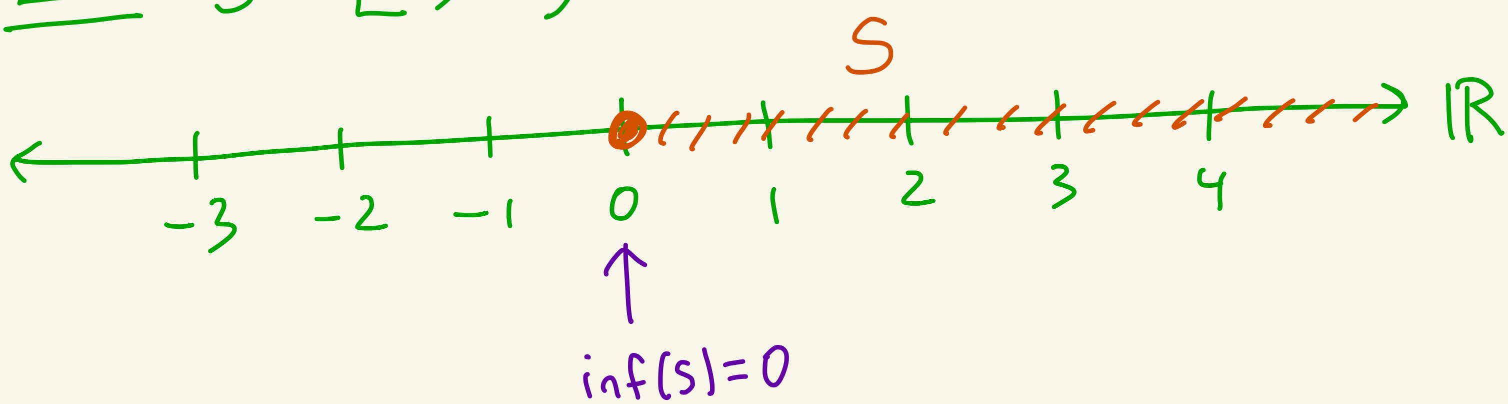
Def: Let $S \subseteq \mathbb{R}$ where S is non-empty.

- We say that b is an <u>upper bound for S</u> if $x \le b$ for all $x \in S$.
  If there exists an upper bound for S then we say that S is <u>bounded from above</u>.
- If b is an upper bound for S and $b \le c$ for all other upper bounds c of S, then b is called the <u>least upper bound for S</u>, or <u>supremum of S</u>, and we write $b = \sup(S)$
- We say that b is a <u>lower bound for S</u> if $b \le x$ for all $x \in S$.
- If there exists a lower bound for S then we say that S is <u>bounded from below</u>.
- If b is a lower bound for S and $c \le b$ for all other lower bounds c of S, then b is called the <u>greatest lower bound for S</u>, or <u>infimum of S</u>, and we write $b = \inf(S)$

Ex: $S=(2,5]\cup\{1\}$



Ex: $S=[0,\infty)$



S is bounded from below with $\inf(S)=0$.
S has no upper bound, so $\sup(S)$ does not exist.

Theorem: Let $S\subseteq\mathbb{R}$ with $S\neq\emptyset$.
If $\sup(S)$ exists then it is unique.
If $\inf(S)$ exists then it is unique.

proof: HW $\boxed{/\!/\!/}$

The Completeness Axiom for $\mathbb{R}$

Let $S \subseteq \mathbb{R}$ be non-empty.

If S is bounded from above, then $\sup(S)$ exists in $\mathbb{R}$.

If S is bounded from below, then $\inf(S)$ exists in $\mathbb{R}$.

You only have to assume this part of the completeness axiom. The second part about inf's can be proven to follow from it. See the proof at the end of these notes.

Ex: $S = [0,2)$

S is bounded from above.

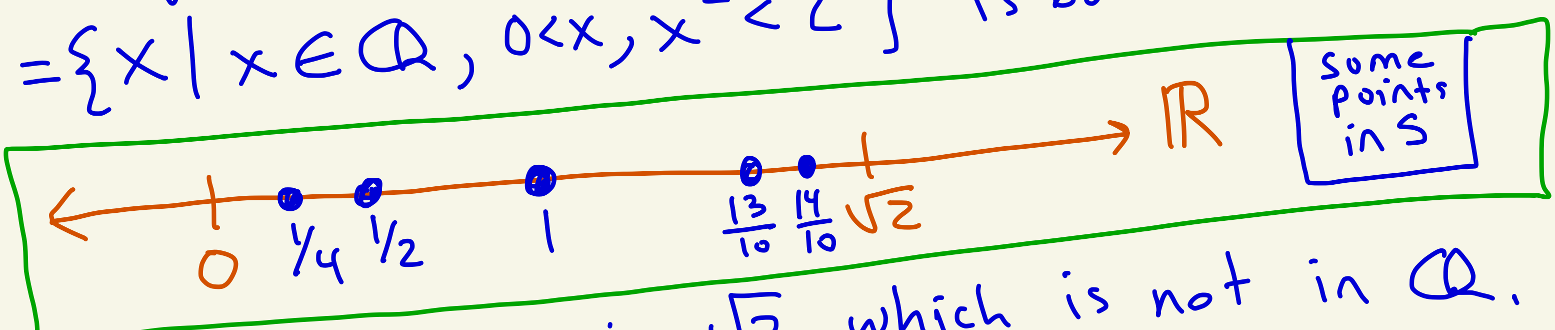
$\sup(S) = 2$ is in $\mathbb{R}$

S is bounded from below

$\inf(S) = 0$ is in $\mathbb{R}$

Note: $\mathbb{Q}$ is an ordered field but it doesn't satisfy the completeness axiom.

$S = \{x \mid x \in \mathbb{Q}, 0 < x, x^2 < 2\}$ is bounded from above



but the supremum is $\sqrt{2}$ which is not in $\mathbb{Q}$.

Theorem (Archimedean property)

Let x be a real number.
Then there exists $n \in \mathbb{N}$ with $x < n$

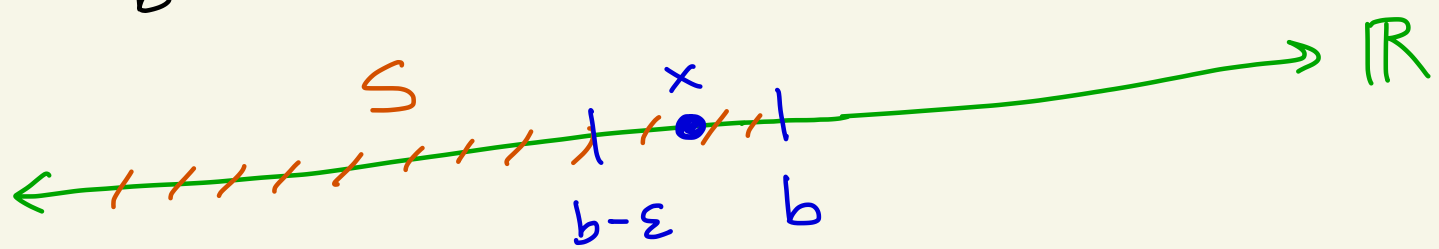
Ex: $x = 20\pi \approx 62.83$
$n = 63$

proof:
Suppose there exists $x \in \mathbb{R}$ where $n \leq x$ for all $n \in \mathbb{N}$.
Then $\mathbb{N} \subseteq \mathbb{R}$ is bounded from above.
By the completeness axiom $\alpha = \sup(\mathbb{N})$ exists.
Then, $\alpha - 1$ is not an upper bound for $\mathbb{N}$.
So there exists $n \in \mathbb{N}$ with $\alpha - 1 < n$.
But then $n+1 \in \mathbb{N}$ and $\alpha < n+1$.
This contradicts $\alpha = \sup(\mathbb{N})$. $\blacksquare$

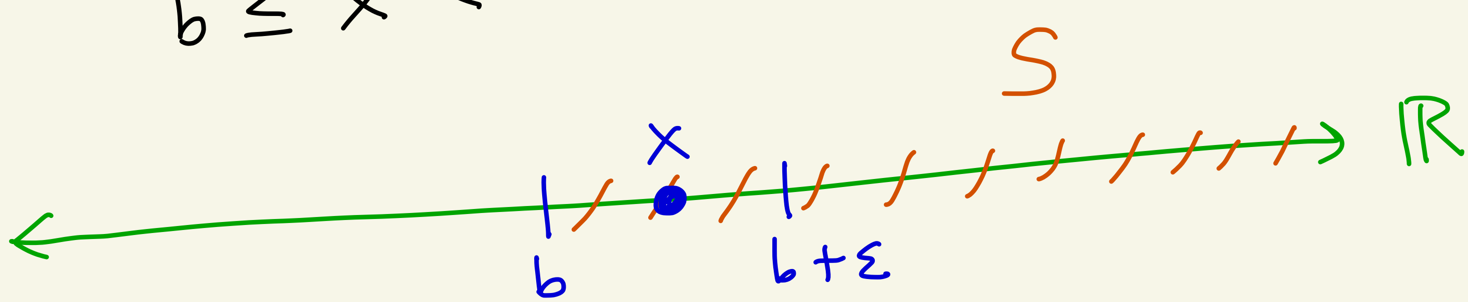
Theorem: (Inf-sup Theorem)

Let $S \subseteq \mathbb{R}$ be non-empty.

(a) Suppose b is an upper bound for S. Then, b is the supremum of S if and only if for every $\varepsilon > 0$ there exists $x \in S$ satisfying
$$b - \varepsilon < x \leq b.$$



(b) Suppose b is a lower bound for S. Then, b is the infimum of S if and only if for every $\varepsilon > 0$ there exists $x \in S$ satisfying
$$b \leq x < b + \varepsilon$$



proof of (a):

($\Rightarrow$) Let b be an upper bound for S and suppose b is the supremum of S.

Let $\varepsilon > 0$.

Since $b - \varepsilon < b$ and b is the least upper bound for S we know that $b - \varepsilon$ is not an upper bound for S.

Thus there exists $x \in S$ with $b - \varepsilon < x$.

Since b is an upper bound for S we also have $x \le b$.

Thus, $b - \varepsilon < x \le b$.

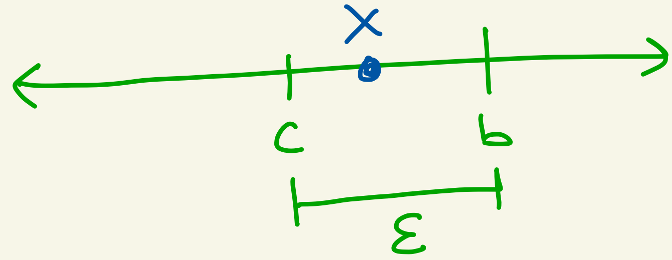
($\Leftarrow$) Let b be an upper bound for S and suppose that for every $\varepsilon > 0$ there exists $x \in S$ with $b - \varepsilon < x \le b$.

Let's show that b is the supremum of S.

Let c be any real number with $c < b$.

Let's show that c can't be an upper bound for S.

Let $\varepsilon = b - c > 0$.



By our assumption there exists $x \in S$ with $b - \varepsilon < x \leq b$

So, $c < x$ with $x \in S$.

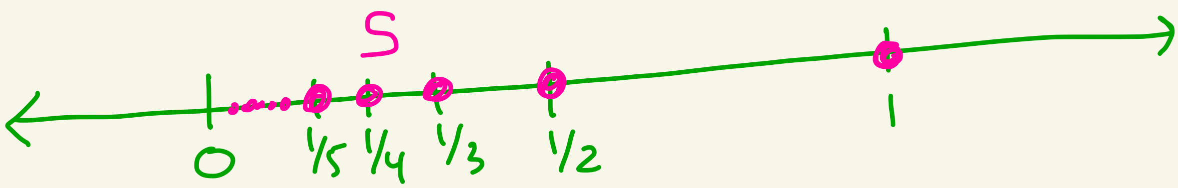
Thus, c is not an upper bound for S.

There fore, b is the least upper bound for S.

The proof of (b) is similar to (a).

∎

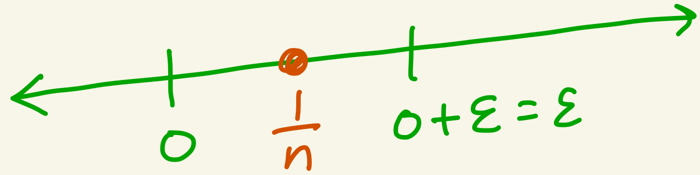
Ex: Let $S = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$



We know that 0 is a lower bound for S since $0 < \frac{1}{n}$ for all $n \in \mathbb{N}$.

Let's show that $0 = \inf(S)$.

Let $\varepsilon > 0$.



We need to find $x \in S$ with $0 \leq x < 0 + \varepsilon$.

Pick $n_0 \in \mathbb{N}$ with $n_0 > \frac{1}{\varepsilon}$.

Then, $\frac{1}{n_0} < \varepsilon$

Set $x = \frac{1}{n_0}$.

Then, $x \in S$ and $0 \leq x < 0 + \varepsilon$.

Thus, by the inf/sup theorem we have that $0 = \inf(S)$.

Def: Let $x \in \mathbb{R}$.

The **absolute value** of x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Ex: $|17.23| = 17.23$

$|-5.1| = -(-5.1) = 5.1$

Theorem:

Let $a, b, c \in \mathbb{R}$ with $c > 0$.

Then:

1. $|ab| = |a| \cdot |b|$
2. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ if $b \neq 0$
3. $|a| \leq c$ iff $-c \leq a \leq c$
4. $|a| < c$ iff $-c < a < c$
5. (Triangle inequality) $|a+b| \leq |a| + |b|$
6. $||a| - |b|| \leq |a-b|$

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proof:

1 / 2 HW

3

($\Rightarrow$) Suppose $|a| \leq c$.

If $a < 0$, then $a < -a = |a| \leq c$.

If $a \geq 0$, then $-a \leq a = |a| \leq c$.

In both cases we get $a \leq c$ and $-a \leq c$.

Thus, $\underbrace{-c \leq a}_{\text{comes from } -a \leq c} \leq c$.

($\Leftarrow$) Suppose $-c \leq a \leq c$.
Then, $-c \leq a$ and $a \leq c$.
So, $-a \leq c$ and $a \leq c$.
Thus, $|a| \leq c$.

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④ Similar proof to part 3.

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⑤ Note first that if $x \in \mathbb{R}$, then $|x| \leq |x|$. Thus, by taking $c = |x|$ in part 3, we get $-|x| \leq x \leq |x|$.

Thus given $a, b \in \mathbb{R}$ we have
$$-|a| \leq a \leq |a| \quad \text{and} \quad -|b| \leq b \leq |b|$$
Adding these two inequalities gives
$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$
Thus, by part 3 again we have
$$|a+b| \leq |a| + |b|.$$

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⑥ HW.

$\blacksquare$

A frequently used fact is this:

Corollary:

Let $x, y, \varepsilon \in \mathbb{R}$ with $\varepsilon > 0$.

Then:

$|x-y| < \varepsilon$ iff $y - \varepsilon < x < y + \varepsilon$

proof:

$|x-y| < \varepsilon$

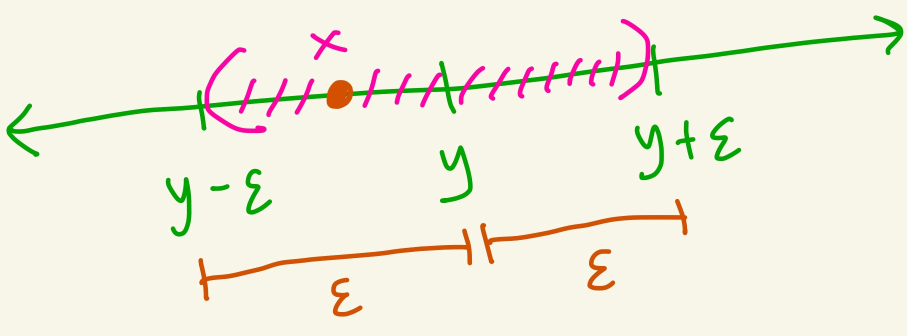
part 3 of above theorem

iff $-\varepsilon < x - y < \varepsilon$

iff $y - \varepsilon < x < y + \varepsilon$

$\blacksquare$

picture of corollary:



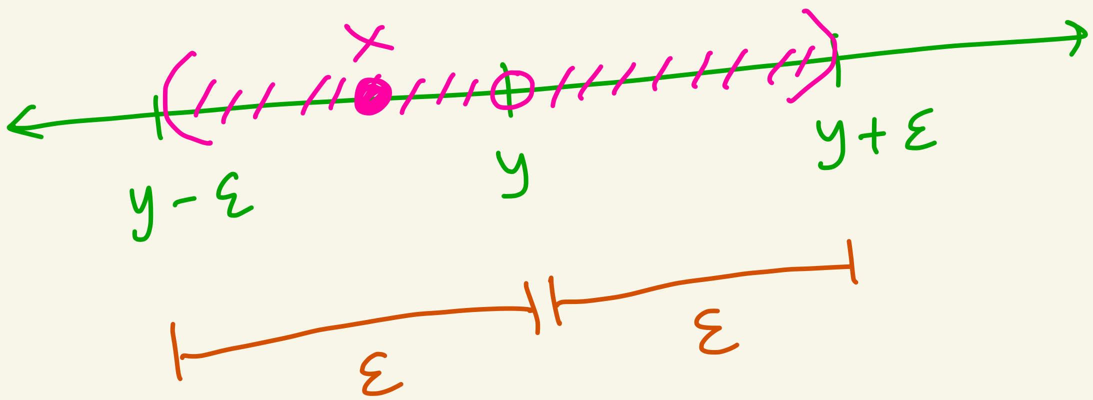
$|x-y| < \varepsilon$ means x & y are within ε distance of each other

We will frequently write:

$$0 < |x-y| < \varepsilon$$

Note: $0 < |x-y|$ means $x \neq y$.

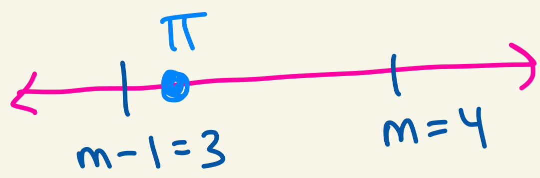
Thus, $0 < |x-y| < \varepsilon$ looks like this:



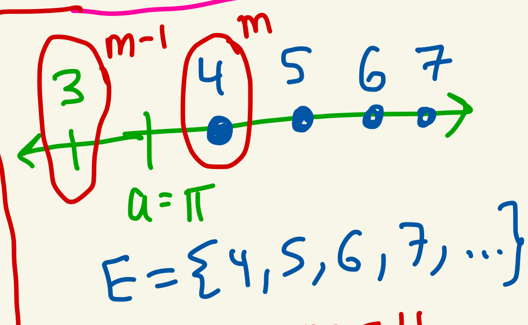
Lemma: Let $a > 0$ be a real numbes. Then there exists a natural number $m \in \mathbb{N}$ with $m-1 \leq a < m$

Ex: $a = \pi \approx 3.14159\ldots$

$m = 4$



proof:



Let $E = \{ n \mid n \in \mathbb{N}, a < n \}$.

We know that $E \neq \emptyset$ by the Archimedean property.

Let m be the least element of E, which exists because $E \subseteq \mathbb{N}$.

Then, $m-1 \notin E$.

Since $m-1 \notin E$ we know $m-1 \leq a$.

Since $m \in E$ we know $a < m$.

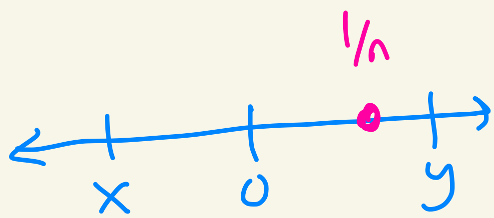
Thus, $m-1 \leq a < m$ $\blacksquare$

Theorem: ($\mathbb{Q}$ is dense in $\mathbb{R}$)
Given $x, y \in \mathbb{R}$ with $x < y$, there exists $r \in \mathbb{Q}$ with $x < r < y$.

Ex: $x = \sqrt{2}$, $y = 2$
$r = \frac{7}{4} \approx 1{,}75$

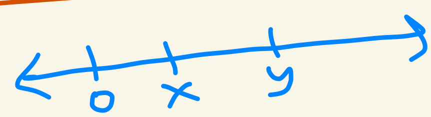
Proof:

Case 1: Suppose $x \leq 0 < y$.
Pick a natural number n with $n > y$.



Then, $x \leq 0 < \frac{1}{n} < y$.

Case 2: Suppose $0 < x < y$.



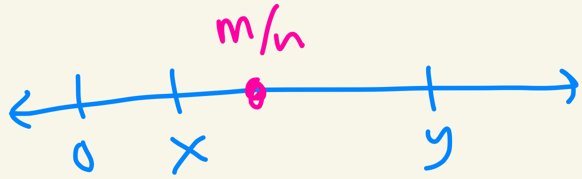
Note that $y - x > 0$.
Pick a natural number n with $n > \frac{1}{y-x}$
Then, $\frac{1}{n} < y - x$.
So, $1 < ny - nx$
Thus, $nx + 1 < ny$
By the previous lemma there exists a natural number $m \in \mathbb{N}$ with $m - 1 \leq nx < m$.

Then, $m \leq nx + 1$.

So, $m \leq nx + 1 < ny$.

In summary, $nx < m < ny$.

Thus, $x < \frac{m}{n} < y$.



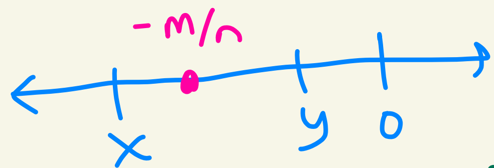
Set $r = \frac{m}{n}$.

<u>case 3:</u> Suppose $x < y < 0$.

Then, $0 < -y < -x$.

Apply case 2 to get $-y < \frac{m}{n} < -x$.

Then, $y < \frac{-m}{n} < x$.



Set $r = \frac{-m}{n}$. $\blacksquare$

Theorem: Given $a, b \in \mathbb{R}$ with $a < b$ there exists an irrational number x with $a < x < b$.

[The irrational numbers are $\mathbb{R} - \mathbb{Q}$
In 2450/3450 you show for example
that $\sqrt{2}$ is irrational.]

proof:
From 3450, the set (a,b) is uncountable.
Since $\mathbb{Q}$ is countable, we know $\mathbb{Q} \cap (a,b)$ is countable since its contained in $\mathbb{Q}$.
Thus, $(a,b) - \mathbb{Q} \cap (a,b) \neq \emptyset$.
Let $x \in (a,b) - \mathbb{Q} \cap (a,b)$.
Then x is irrational and $a < x < b$. $\blacksquare$

This next part is optional.
It shows we only had to assume
half of the completeness axiom

Suppose we only assume the following part of the completeness axiom for $\mathbb{R}$:

If A is a non-empty subset of $\mathbb{R}$ that is bounded from above, then $\sup(A)$ exists in $\mathbb{R}$

We now show that this will imply the following:

If B is a non-empty subset of $\mathbb{R}$ that is bounded from below, then $\inf(B)$ exists in $\mathbb{R}$

Proof: (from Rudin's book)

Let $B \subseteq \mathbb{R}$ with $B \neq \emptyset$.

Suppose B is bounded from below.

Let

$$L = \{ y \mid y \in \mathbb{R} \text{ and } y \text{ is a lower bound for } B \}.$$

By assumption $L \neq \emptyset$.

Note that if $x \in B$ then $y \leq x$ for all $y \in L$.

Since $B \neq \emptyset$ this implies that every x in B is an upper bound for L.

Since $L \neq \phi$ and bounded from above we know that $\alpha = \sup(L)$ exists in $\mathbb{R}$.

We will show that α is the infimum of B

If $\gamma < \alpha$ then by def of supremum we must have γ is not an upper bound for L.

So, if $\gamma < \alpha$ then $\gamma \notin B$.

Thus, $\alpha \leq x$ for all $x \in B$.

Therefore, α is a lower bound for B and $\alpha \in L$.

Why is α the greatest lower bound for B?

Suppose β satisfies $\alpha < \beta$.

Then, since α is the supremum of L we must have that $\beta \notin L$.

That is, if $\alpha < \beta$ then β is not a lower bound for B.

So all lower bounds β for B must satisfy $\beta \leq \alpha$.

Therefore $\alpha = \inf(B)$. $\blacksquare$